Spreading Wave Packets

Goal
We look at wave functions for moving individual electrons and describe the probabilities for its location. We will see how the amount of spreading of the wave function depends on the initial uncertainty in momentum.

Prerequisite
It Was Probably Heisenberg

The wave function for a single electron, also called a wave packet, is created by combining many wave functions with different wavelengths, where the wavelength is determined by the electrons momentum. This wave packet provides information about both the location and momentum of the object. However, it cannot give us exact values for either quantity. We say that the wave function leaves us with an *uncertainty in position* and an *uncertainty in momentum* for the electron.

Now, we wish to use this wave packet approach to look at how our knowledge of an electron's position and momentum changes over time. To obtain as complete a picture as possible we need to solve the time dependent Schrödinger Equation — a task beyond the scope of this effort.

Instead, we will allow a computer program to show us graphical solutions. We will then interpret these solutions. Begin by starting the *Quantum Motion* program. With this program you can vary the

- Type of potential energy including creating a double potential,
- Height and width of the potential energy;
- Energy of the Gaussian wave packet;
- Starting position of the wave packet; and

- Width of the wave packets.

As we saw in the previous activity varying the width of the wave packet changes the uncertainty in position, Δx. According to Heisenberg's Uncertainty Principle we also change the uncertainty in momentum, Δp.

Motion of a Non-Interacting Particle

For our present purposes we will consider an electron that is traveling in empty space and not interacting. So, set the potential energy to zero.

Now, start the animation. Describe how the wave packet changes as it moves across the screen.

How does the probability density change?

Compare the uncertainty in position, Δx, as the wave packet starts with Δx near the right of the screen. How does Δx change? Explain how you reached your answer.

We can understand this result by returning to the momenta involved. We start with a wave function as in Figure 1. The electron can have many different momenta. (Remember we needed all those momenta to construct its wave function.) Of course, it also has a range of possible locations. So, we don't quite know where it is, and we don't quite know how fast it is moving — an uncertainty in position; an uncertainty in momentum.

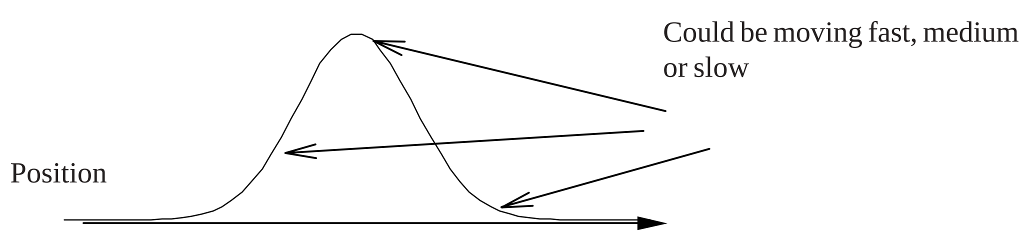


Figure 1: When we start, we have an uncertainty in both position and momentum.

A short time later we look to see where the electron is. We expect to still have an uncertainty in position because there was such an uncertainty when we started. In addition, the electron had an uncertainty in its momentum at the start. This uncertainty means that we do not know exactly how fast the electron is moving. The location of an object at a later time depends on both its starting point and its speed. In the case of the electron both the start and the momentum have uncertainties. As time passes the initial uncertainty in position is compounded by the uncertainty in momentum. So, as an electron moves the uncertainty in position increases.

This change in uncertainty is represented in *Quantum Motion* by a change in the wave function. As time passes, the wave function spreads out. This spreading indicates a wider range of positions is probable as the electron moves.

Changes in Δx

As the electron moves the locations of highest probability became smaller, and the range of locations with medium and low probability became greater. (See Figure 2).

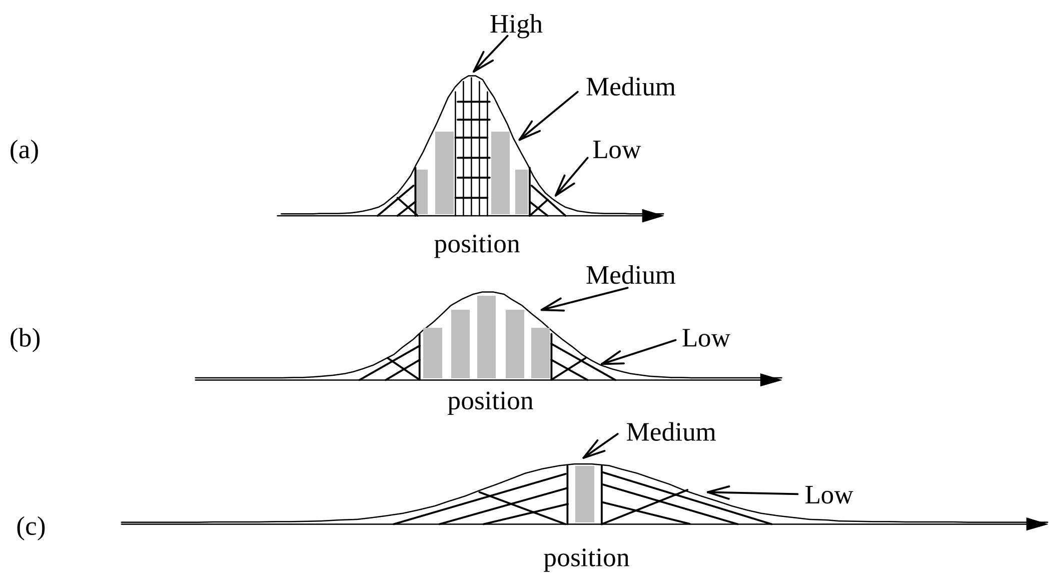
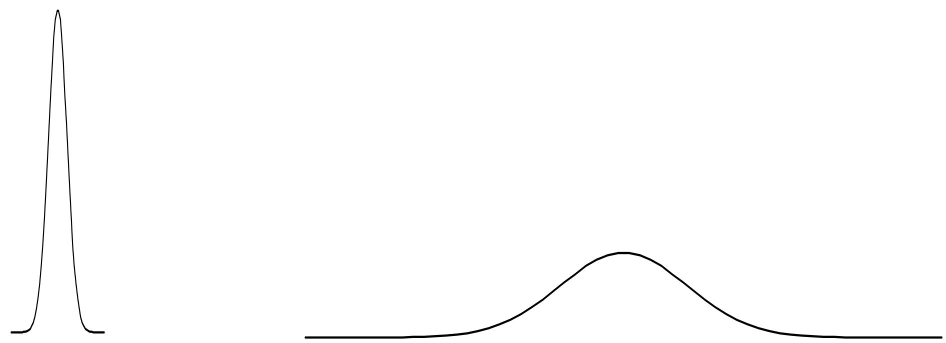


Figure 2: The wave function spreads out of time. Thus, the range of probability locations increases.

Now, explore how change in Δx depends on the initial value of Δx. Set up two different wave functions with shapes similar to those in Figure 3.



(a) (b)

Figure 3: Wave functions that (a) represent an object restricted to a very small region of space and (b) not very restricted.

Start each wave packet, then stop it a short time later. For which wave packet is Δx greater after a short time?

Explain your result in terms of the Uncertainty Principle and the discussion above.

As you see the uncertainty in position increases more rapidly when the initial Δx is smaller. The rate of the uncertainty in positions depends on the uncertainty at the start. Wave function (a) in Figure 3 is restricted to a very small region of space, while wave function (b) is not as restricted. The uncertainty in momentum is larger for wave packet (a). Thus, the change in Δx is greater for (a) than for (b). A very small uncertainty in position at the beginning results in a very rapid change in the wave function. A greater uncertainty at the start means the change in uncertainty is not so great. How well we know the position at one location determines how well we can know it later.

Mathematical Approach

The spreading of the wave packet makes sense in terms of the Uncertainty Principle. Suppose a particle has an uncertainty in position of Δx_0 at t=0. The momentum at t=0 is p_0. The uncertainty in momentum at this time is at least

$$\Delta p_0 = \frac{\hbar}{\Delta x_0} \qquad (1)$$

Thus, we can determine the uncertainty in speed as

$$\Delta v_0 = \frac{\Delta p_0}{m} = \frac{\hbar}{m \Delta x_0} \qquad (2)$$

From this we can determine the uncertainty in position and at a later time.

$$\Delta x = \frac{\hbar}{m \Delta x_0} t \qquad (3)$$

Thus, we see that the uncertainty in position at any later time is inversely proportional to the initial position. The better we know it now, the worse we will know it later. Hopefully this principle does not apply to your knowledge of quantum mechanics.