Name Partners Date

It Was Probably Heisenberg

Goal

We will use our knowledge of wave functions to create wave packets to describe an electron. We will discover that the wave nature of the electron actually prevents us from making exact, simultaneous measurements of its position and momentum. This result is known as Heisenberg's Uncertainty Principle, which leads us to some very interesting philosophical issues related to the nature of matter. Finally, we will speculate about the value of the Heisenberg Compensator in the Star Trek Transporter.

Introduction

When we describe the motion or position of individual particles we have some knowledge of what limits the probability to a small region of space. A measurement for example, could state that an electron is located at $A \pm 0.1$ nm. Figure 1 shows one possibility for the probability density related to such information.

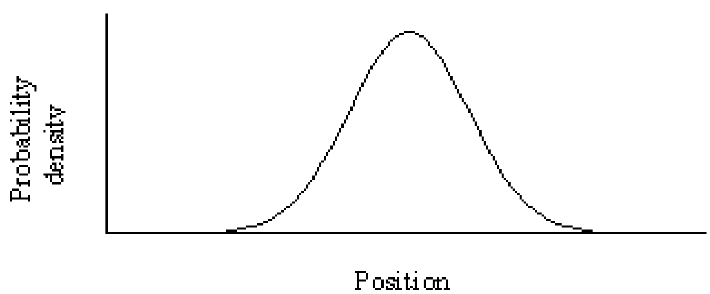

Figure 1: Typical shape for the probability density of a localized *electron.*

To obtain this probability density, we must construct a wave function that, when squared, gives a probability density with one maximum, decreases sharply away from that maximum, and then is zero at relatively large distances from where we expect to find the electron. A wave function such as this will represent a single electron which is most probably located near the maximum of the wave function.

A. Adding Waves

The ability of waves to interfere both constructively and destructively enables us to combine several different waves to create *waveforms* with a variety of different shapes. Even waveforms such as the disturbance on the rope can be constructed by adding simple waves together. Our goal will be to combine simple waves to produce a wave function that, when squared, gives a smooth probability density graph similar to Figure 1.

To see how we might create such a wave look at the addition of waves shown in Figure 2.

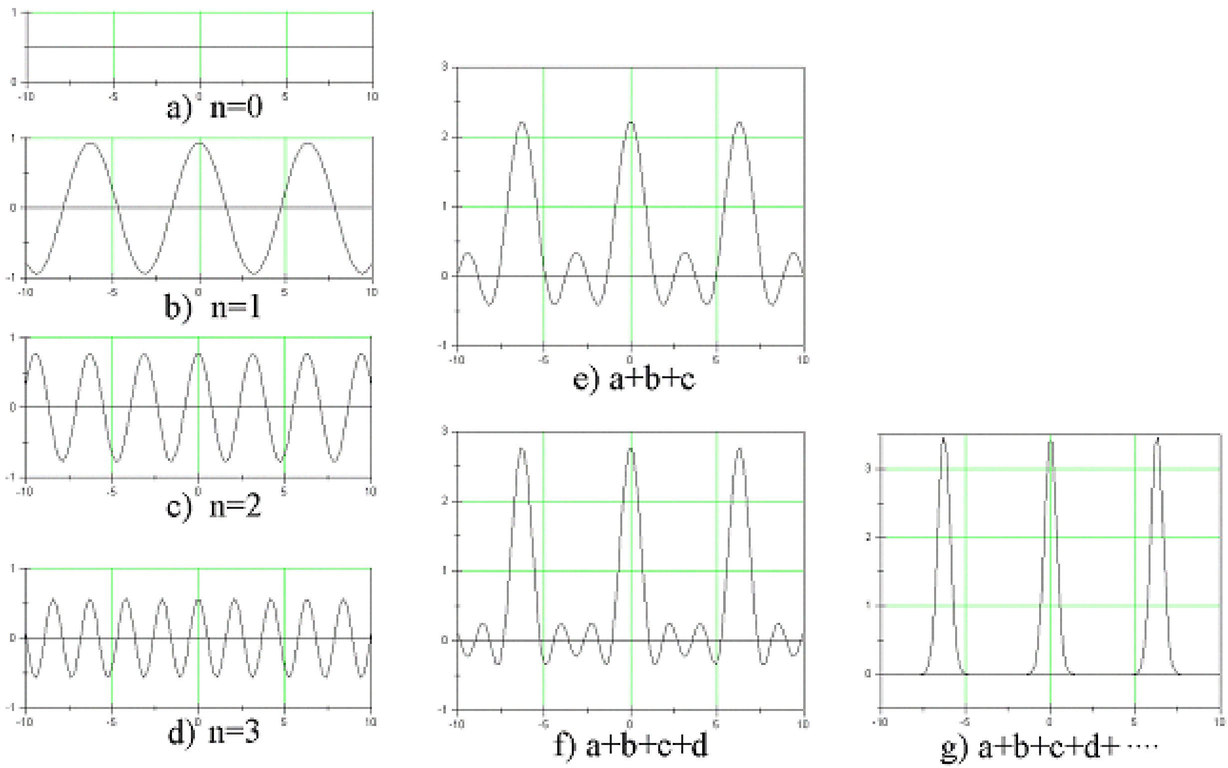

Figure 2: Adding waves a), b), and c) produces the wave shown in e). Adding in another wave, shown in figure d) results in the wave shown in figure f). Figure g) shows the wave packet resulting from the addition of ten waves, each with a different amplitude and wavelength.

In Figure 2 waves with different amplitudes and wavelengths are added to produce a series of wave packets. By adding hundreds of waves with carefully selected wavelengths and amplitudes we can come very close to the form in Figure 1. Each of the individual waves in Figure 2 has the form,

$$e^{-[n^2/2\sigma]} \cdot Cos(n \cdot x) \qquad (1)$$

with different values of *n* for every wave. As you can see from the figure, as *n* increases the wavelength decreases due to the *Cos(n x)* term. From the de Broglie relation we know that wavelength is related to momentum. So, when we add wave functions of different wavelengths, we are adding wave functions representing objects with different momenta.

The waveform in Figure 1 is produced from a *Gaussian* or bell shaped distribution of momenta. The factor σ in equation (1) is related to the width of the momenta distribution. We will return to this factor later after you have had a chance to create some of your own wave forms.

To create wave functions such as this one use the *Wave Packet Explorer*. This program allows you to add wave functions of different momenta (wavelengths) rather easily.

Start the creation of a wave function by clicking in the upper left window (*Amplitude vs. Momentum* graph).

A vertical line represents the amplitude and momentum of a sine wave. The wave's amplitude is proportional to the length of the line, and its momentum is the value on the horizontal axis. The wavelength is calculated from using the de Broglie relation. The wave function appears in the window directly below this line.

The program allows the following actions:

- add a wave by clicking in the amplitude-momentum graph.
- look at an individual component by clicking at the top of its amplitude-momentum bar.
- change either the momentum or amplitude by dragging the bar horizontally or vertically.

Create a wave function by using 3-6 sine waves of different momenta.

A-1. How does the resulting wave function change as you increase the number of sine waves?

A-2. Sketch both your momentum distribution and resulting wave function below.

A-3. Now, find two other groups with wave functions that look somewhat different from yours. Sketch their momentum distributions and wave functions below.

A-4. Describe how the momentum distributions contribute to the different forms for the wave functions.

A-5. You can also add a large number of waves. With the shift key pressed, click and drag the mouse in some pattern across the top graph. Sketch the momentum distribution and resulting wave functions below.

A-6. Compare your resulting wave functions with others in the class. Again, find a couple that are different from yours. Describe how their momentum distribution is different from yours.

Wave functions which are restricted to a small region of space are called wave packets.

B. Wave Packets and Probabilities

Now, try an amplitude-momentum distribution similar to the one in Figure 3.

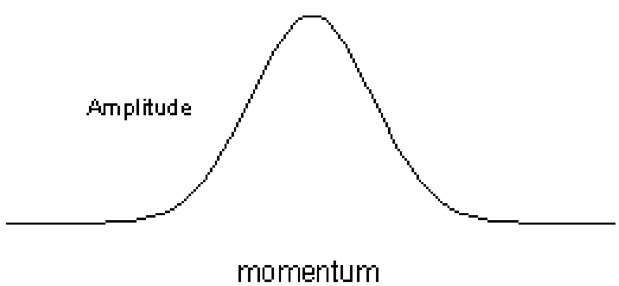

Figure 3: An amplitude-momentum distribution that gives a wave function and probability density similar to the one in Figure 1.

B-1. Sketch the position wave packet and probability density below.

B-2. Does this wave packet have a large or small range of momenta associated with it? Explain your answer.

Now, try a wave packet and probability density which has a very narrow range of momenta. Create it by dragging over a small range.

B-3. Sketch the position wave packet and probability density below.

B-4. How is the probability interpretation different from the previous wave function?

As you can see from these two examples, the range of momenta affects the width of the central peak of the wave packet and, in turn, affects the probability density. A wide range of momenta results in a strong central peak, corresponding to a high probability of the electron being in a very narrow region of space, while a small range of momenta gives us a broad "peak".

We see that we can create a wave packet that represents an electron in a small region of space. To do so we must add together many different sine waves. Each sine wave in the summation has a different wavelength. Because wavelength is related to momentum, each sine wave represents a different possible momentum for the electron.

Thus, we need many different momenta to create a wave function for one electron which is localized in space. We interpret this result to mean that these localized electrons have a probability of having many different momenta. The probability of each momentum is related to the amplitude of the sine wave with that momentum (Figure 4).

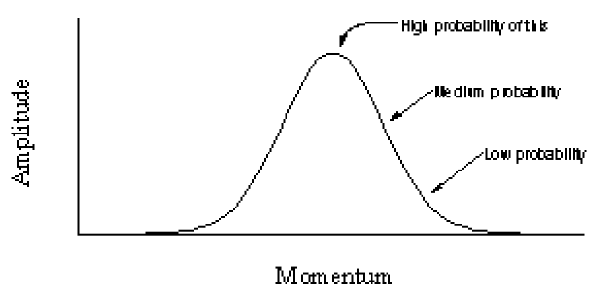

Figure 4: The probability of each momentum depends on the amplitude on the amplitude-momentum graph.

We must make a similar statement about position. Even when we add waves with many different momenta, we still get a wave function similar to Figure 5.

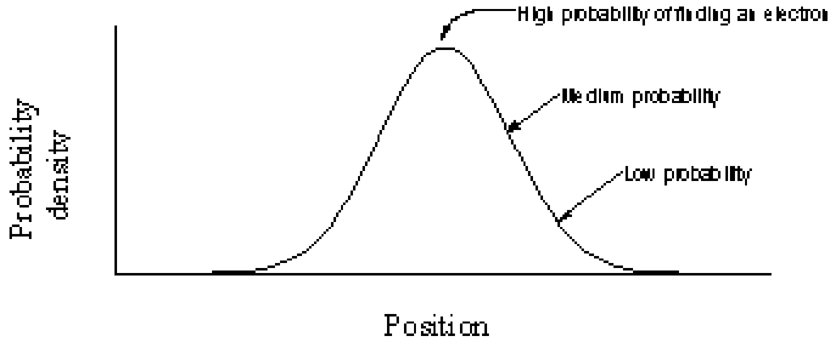

Figure 5: The relative probability of finding the electron at various locations for a localized wave function.

B-5. Compare these results to a wave function which contains only *one* momentum. Sketch the probability density for a single momentum wave function.

B-6. Describe the probability of finding an electron described by this wave function in various locations in space.

C. Relation Between Position and Momentum Uncertainties

We can create a wave function that localizes the electron to a small region of space. In doing so we increase the possible range of momenta that the electron can have. We must talk about probabilities for both the electron's location and its momentum. This process leads to uncertainties in our knowledge about both the electron's position and its momentum.

We will now investigate wave packets that have different uncertainties in position and momentum. To begin, use the *Wave Packet Explorer* to create wave functions with different uncertainties in momentum. Your instructor will select a different wave function from Figure 6 for each group to create.

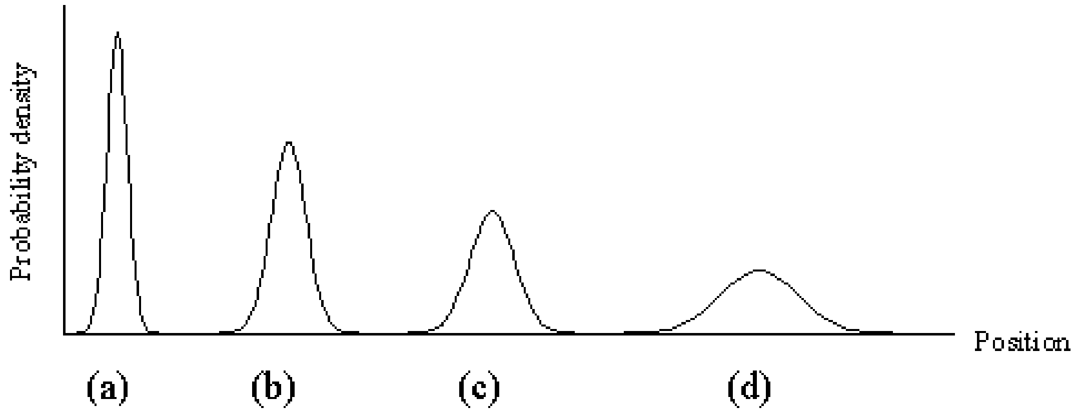

Figure 6: Wave packets with different uncertainties in momentum.

C-1. Compare the result of all groups. Which one needed to include the largest number of momenta and which one needed the smallest number of momenta?

C-2. How is the uncertainty in position related to the uncertainty in momentum?

D. Heisenberg's Uncertainty Principle

The exercise you have just completed illustrates the relationship between uncertainties in position and momentum. Creating a wave packet with a small uncertainty in position (Figure 6a) requires a large number of momenta. Thus, there is a large uncertainty in the momentum. If we decrease the uncertainty in momentum by using fewer waves of different momentum to create the wave packet, we end up increasing the uncertainty in position. This exercise indicates that as one of the uncertainties increases, the other decreases. This conclusion is built into the wave nature of matter. It does not depend on our measurement instruments. (We have not discussed measurement here — only creating wave functions.)

The mutual dependence of the uncertainty in position and the uncertainty in momentum was first stated by Werner Heisenberg. His statement is known as *Heisenberg's Uncertainty Principle*. It says that the uncertainty in position and the uncertainty in momentum are closely related. If one decreases, the other increases by the same factor. Mathematically, the Uncertainty Principle is written as:

$$\Delta x \cdot \Delta p = \text{constant} \qquad (2)$$

where Δx is the uncertainty in position, Δp is the uncertainty in momentum.

This principle states that we can never know both the exact position and the exact momentum of an electron at the same time. The best we could do is that the uncertainties are related by this equation. We would get these results only with perfect measuring instruments. We can always do worse. For this reason, Heisenberg's Uncertainty Principle is usually stated as an inequality:

$$\Delta x \cdot \Delta p \geq \frac{h}{2\pi} \qquad (3)$$

where h is Planck's constant.

D-1. For example, suppose we establish the position of an electron precisely to within a tenth of a nanometer (0.0000000001 m). Using $\Delta x = 10^{-10}$ m, what would be the *minimum* uncertainty in the electron's momentum (Δp)? (The value of Planck's constant is $h = 6.63 \times 10^{-34}$ J·s)

D-2. With this uncertainty in momentum, what would be the corresponding uncertainty in the *speed* of the electron? (The electron's *mass* is 9.11×10^{-31} kg.) Recall that momentum = mass x velocity.

D-3. A reasonable speed for an electron might be around 10^6 m/s. Is the uncertainty in speed that you calculated significant when compared to this speed? Explain.

E. Deriving Heisenberg's Principle

We begin with a packet of waves created with individual waves having a momentum distribution that follows a bell shaped or *Gaussian* distribution. Using this wave packet we will obtain uncertainties in position and momentum to arrive at Heisenberg's Uncertainty Principle.

Here to make the notation a little easier, instead of using momentum we will use the closely related concept of wave number, *k*. The wave number is just the momentum multiplied by a constant $k=p(2\pi/h)$. To add waves with different wavelengths we now add waves with different wave numbers. To obtain a Gaussian distribution we must use an infinite number of waves which is represented by an integral. Thus, we take an infinite sum of waves with infinitesimally close wave numbers *k* by integrating

$$F_c(x,t) = F_0 \int_{-\infty}^{+\infty} dk \cdot f(k) \cdot e^{-i(\omega t - kx)} \qquad (4)$$

$F_c(x,t)$ is the amplitude of the wave form at x and t. The constants in $f(k)$ below are chosen so that the integral normalizes to one and F_0 represents the maximum value of $F_c(x,t)$. The function $f(k)$ represents the amplitude of the wave with wave number k. For a Gaussian distribution

$$f(k) = \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{\frac{-(k-k_0)^2}{2\sigma^2}} \qquad (5)$$

This $f(k)$ distribution of waves is a bell shaped curve centered at $k=k_0$ with a spread related to σ as shown in Figure 7. (The quantity σ is the standard deviation of the wave form.) We use σ as an indication of the uncertainty in k. Thus,

6)

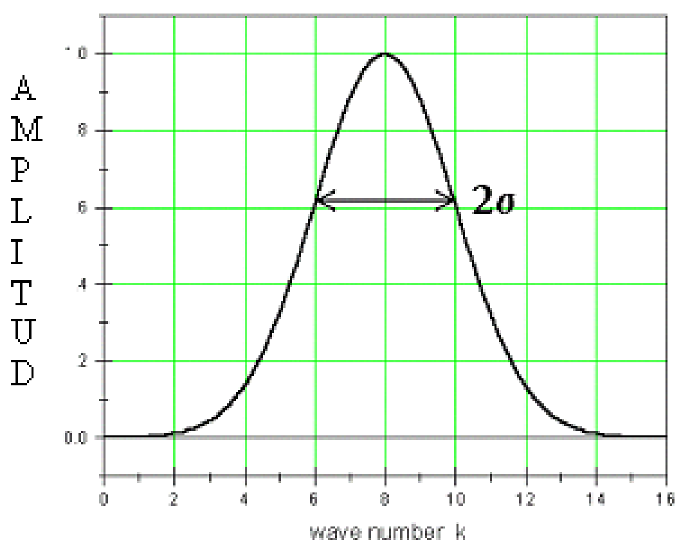

Figure 7: A Gaussian with σ=2 and k_0=8.

If we perform the integral in Equation 4 and keep only the real portion, we get a wave packet where

$$F(x,t) = F_0 e^{-(\sigma^2/2)\cdot(ct-x)^2} \cos(\omega_0 t - k_0 x) \qquad (7)$$

This wave form is a *Gaussian* wave in position space centered at $ct=x$. The width of this wave is $1/\sigma$. The width represents the spread in position and we take this to be the uncertainty in position

$$\Delta x = \frac{1}{\sigma} \tag{8}$$

Since the wave number is related to the momentum by the relation $k = p/\hbar$, then $\Delta k = \Delta p/\hbar$ when $\hbar = h/2\pi$. Combining this with equations (6) and (8) D*k*D*x*=s/s=1 we obtain Heisenberg's uncertainty relation

$$\Delta x \Delta p = h/2\pi. \tag{9}$$

Again, this is a minimum value for the Uncertainty Relation. Any actual measurement will result in:

$$\Delta x \Delta p \geq \hbar \tag{10}$$

F. The Uncertainty Principal and Diffraction

We now return to the experiment that introduced us to the wave nature of matter, the diffraction of an electron. On the computer, start the *Single Slit* program in the *Diffraction Suite*. With this program we can simulate the behavior of electrons as they pass through a small opening.

F-1. Using electrons create a diffraction pattern. Sketch it below and record the slit width and energy.

F-2. For the discussion, the x direction will be parallel to the screen where the electrons stop and y will be perpendicular to it. Think about the electron as it passes through the slit. How well do you know its x position? Explain your answer.

F-3. Keep your pattern and repeat the experiment at a rate of one particle per second. Stop when you have about ten electrons on the screen. To reach the screen at their present locations each of these electrons had velocity in the x and y directions. For each electron draw a vector to indicate its velocity in the x direction.

F-4. As you see we have a distribution in velocities. Now, make the slit width very small and run the experiment for a large number of electrons. How do the knowledge of the x position and the distribution of x change?

F-5. Now make the slit width very large. How do the knowledge of the x position and the distribution of x change?

Here we see a result quite similar to the Uncertainty Principle. A narrow slit (small uncertainty in position) gives a wide range of velocities and thus momenta. A wide slit (large uncertainty in position) gives a narrow range of momenta.

We can use these results to derive the Uncertainty Principle. In this case we wish to consider a single slit diffraction as shown in Figure 8.

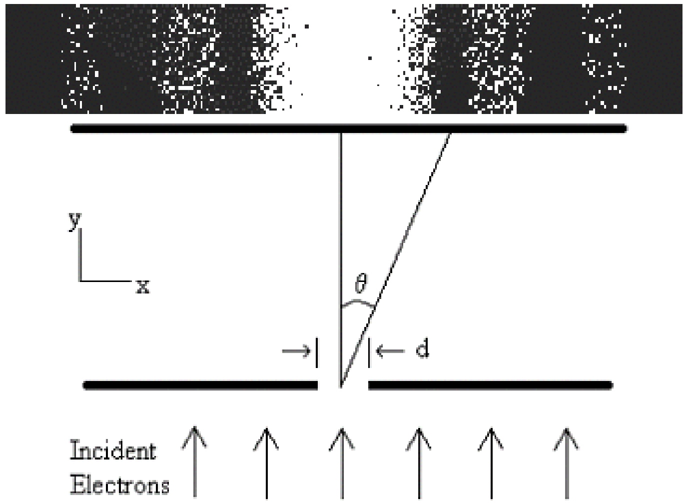

Figure 8: Electrons being diffracted as they pass through a single slit.

When an electron passes through a slit, we know its position up to an uncertainty equal to the width of the slit so we can say $\Delta x = d$. For single slit diffraction the central bright spot is spread out from the center to an angle of

$$\theta \approx \sin\theta = \lambda/d, \tag{11}$$

where the approximation is for small angles. Since we do not know exactly where the electron hits the screen, there is an uncertainty in the velocity (and momentum) in the *x* direction such that

$$\frac{\Delta v_x}{v} \approx \theta \tag{12}$$

We can combine these two equations to obtain

$$\frac{\Delta v_x}{v} \approx \frac{\lambda}{d} \tag{13}$$

Using $\Delta x \approx d$ and the de Broglie relation

$$\frac{\Delta v_x}{v} \approx \frac{h}{p \cdot \Delta x} \tag{14}$$

and

$$\frac{m \cdot \Delta v_x}{m \cdot v} = \frac{\Delta p_x}{p} \approx \frac{h}{p \cdot \Delta x} \tag{15}$$

Finally, we have the Heisenberg Uncertainty Principle $\Delta x \cdot \Delta p \approx \hbar$. Again, we can discuss the minimal possible result and conclude as before.

G. Compensating for Heisenberg

The Heisenberg Uncertainty Principle is applied to all types of matter including that in science fiction. One of the "controversies" in the early StarTrek series involved these uncertainties. SomeTrekkies claimed that the Uncertainty Principle means that transporters would not be feasible. Because of the uncertainties, the transporter would never be able to measure the location of the matter in people well enough to put them together at a distant location exactly as they are. So they could not be beamed elsewhere.

Maybe these uncertainties are so small that they are unimportant in the transporter. For example, suppose a gnat is flying around in the transporter room. When it flies over the transporter pad, we measure its position and momentum. Then try to transport it to a location near somebody's compost pile. We measure the gnat's position only fairly accurately. The uncertainty of its position is within one millimeter (10^{-3} m).

G-1. According to Heisenberg's Uncertainty Principle, what would be the *minimum* uncertainty in the gnat's momentum (Δp)?

So, the gnat's uncertainty position and its momentum are rather small. We could conclude that the Uncertainty Principle is not a problem. But, the transporter does not "work" by beaming whole objects. Instead it measures the locations of all atoms (maybe even all protons, neutrons, and electrons). Then, it disassembles the object and reassembles it elsewhere. Look at the uncertainty calculation for an electron.

G-2. How likely is it that the gnat can even be reassembled intact? Explain.

G-3. How would the Uncertainty Principle affect our ability to transport objects even larger than gnats? (They will have many more atoms.)

The StarTrek writers worked around this problem. According to the *StarTrek: Next GenerationTechnical Manual* each transporter scanner permits "real time derivation of analog quantum state data using a series of dedicated Heisenberg Compensators." This "Heisenberg compensator" somehow allows precise measurement of both position and momentum. They admit that they don't know how such a device would work. However, they recognize that it is needed to overcome these limitations of quantum physics. Its failure also makes for interesting stories. It should be noted that the StarTrek writers have also been insightful and creative enough to invent "subspace." This spatial continuum is hidden within our own, familiar three-dimensional space. It allows the transporter to "beam" people and objects instantaneously from one site to another since, in subspace, travel is not restricted to speeds below the speed of light. Fortunately, fiction is not restricted by the concepts of physics.

Today we lack the technology to construct "Heisenberg compensators." But, we cannot rule out the possibility that something similar may one day be developed. Talk of transporters may sound fantastic, but some current research indicates that "quantum teleportation" (as it is known to scientists) may actually be possible with individual atoms. Researchers at IBM have shown that teleportation of one atom is theoretically possible, but only if the original is destroyed. One atom is far from a person (or even a gnat) and many problems are involved in scaling up. But, it is fun to think about it. (See *Scientific American*, April, 2000)

H. Conclusions

Perhaps the most important aspect of the Uncertainty Principle is its philosophical implications. It states that humans cannot know everything about an object. Even if we try to imagine the "perfect" measuring instruments, we cannot determine the exact position and exact momentum simultaneously. Such measurements are not possible. Thus, the Uncertainty Principle places limitations on humankind's knowledge of everything.

This limitation on our knowledge is inherent in nature. We observe the spectra of atoms. To explain them we conclude that matter behaves as waves. If this description is effective, it must describe individual electrons. This description leads to the Heisenberg Uncertainty Principle. Then, we find we cannot know everything precisely, even if we had perfect measuring instruments.

This lack of ability to know is a major departure from the physics of Newton. In 1787 Pierre Simon LaPlace, a mathematician, considered Newton's Laws carefully. Suppose he knew the initial position and velocity, and the forces on every object in the universe. Then, suppose he could calculate really fast. He concluded that he could, in principle, know all of the future because he knows Newton's Laws. The only thing stopping him is lack of good measurements and slow computations. Quantum mechanics (particularly the Heisenberg Uncertainty Principle) says, "not true." Even with perfect measuring instruments we cannot know positions and velocities accurately enough. The nature of matter, not of measurement, limits our knowledge.

Homework

1. Suppose we determine the position of an electron to within 10 nanometers (10×10^{-9} m). What will be the best possible measurement of its momentum?

2-a. Suppose you are able to determine the speed of a 100 kg pole vaulter to within .04 m/s. Within what accuracy would you be able to determine her position?

2-b. Based on your answer above, do you think that we need to worry about the Uncertainty Principle in measurements that we make in our everyday lives? Explain your answer.