Students' Understanding of Mathematical Integration in Physics Problems Using Graphical and Algebraic Representations

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Abstract

We report on a study to explore students' understanding of integration as an accumulation process in the context of physics problems in graphical and algebraic representations. Twenty students in a calculus-based physics course were interviewed several times during the semester. In these interviews, students solved several physics problems in which information was provided in graphical and algebraic representations. A facilitator interacted with the student to explore his/her reasoning and to provide hints whenever the student was not able to proceed. Here we analyze students' performance on several integration tasks embedded in the interview problems on work and energy. To solve these problems, students had to calculate work done by a force using the concept of integration as the area under the curve in the graphical representation or as the algebraic integral of force function in the algebraic representation. We analyzed students ideas of integration using the theoretical framework of concept projection and found that although most of the students were able to recognize the use of integration, calculate the area under the curve and compute the integral correctly, only one student indicated an understanding of the accumulating process underlying those calculations.

Introduction

Integration is a powerful mathematical tool used in a wide range of physics problems. In most problems in a typical calculus course, students are asked to compute pre-determined integrals. However, in most physics problems, the integrals are not provided. Students have to set up the integrals describing a physical quantity by translating the physics scenarios described in the problem statement into the corresponding mathematical integrals. Hence, physics problems involving integration are usually difficult for students at the introductory physics level, not necessarily because the integrals are difficult to compute, but because they require an understanding of integration as the accumulation of infinitesimally small quantity to get the total quantity. In this study, we analyze students' interaction with a facilitator on several physics problems involving integration to investigate the extent to which students can recognize the use of integration and understand the accumulation process when performing integration.

The theoretical framework for this study is the "transfer in pieces" framework proposed by diSessa and Wagner (2005). Specifically, we employ Wagner's notion of "concept projection" which "refers to a specific combination of knowledge resources and cognitive strategies used by an individual to identify and make use of a concept under a particular contextual condition" (Wagner, 2006, p. 10). The range of contexts across which an individual's concept projection is found to be applicable constitutes the span of that concept projection (Wagner, 2006). In this proposal, we focus our discussion on three sets of interviews completed with 20 students in a calculus-based physics class. The interview problems involve calculating physical quantities from other non-constant quantities described by mathematical functions provided in algebraic and graphical representations. The design of the interviews will be described in the next section.

We investigate the concept projection of integration spanning graphical and algebraic representations of a function. In the graphical representation, the concept projection of integration is the "area under the curve", while in the algebraic representation, the concept projection of integration is the algebraic integral. Understanding these two concept projections in the context of the work-energy theorem requires learners to pull together the common knowledge resource of "integration as accumulation" and the cognitive strategy of "accumulating incremental quantities to obtain the total quantity". The research question we pose in this study is: *To what extent does students' concept projection of integration span the context of work-energy problems that provide information in graphical and algebraic representations?*

Methodology

We conducted individual teaching/learning interviews (Steffe, 1983; Steffe & Thompson, 2000) with 20 students randomly selected from a pool of 102 volunteers enrolled in a first-semester calculus-based physics course. Most of the participants were freshmen or sophomore engineering majors with a high school physics background. Each student was interviewed four times during the semester, each time after students had completed an exam in their course. In three of these interviews (interviews 2, 3 and 4) students were asked to solve problems which required them to apply the concept of integration to solve problems using the work-energy principle. The teaching/learning interview format allows the interviewer to also serve as a facilitator and provide scaffolding in forms of verbal hints and cues to enable the student to solve the problem. The goal of this process, hence, is not merely to probe students' knowledge, but also to probe how they respond to the scaffolding. All interviews were audio- and video-taped and transcribed. Students' written work during the interview was also collected.

In each interview, each student was asked to solve two problems in which parts of the information were provided as mathematical functions in graphical and algebraic representations. The graphical problem is a physics problem in which part of the information was provided as a graph of a function. The algebraic problem is an isomorphic problem in which the same information was provided as an algebraic equation.

As part of a study reported previously, we investigated the interaction effects between these two representations (Nguyen & Rebello, 2009). So, approximately half of the participants were given the graphical problem before the algebraic problem while the other half were given the algebraic problem before the graphical problem. We found that in both cases, better performance was observed in the second problem (Nguyen et al., 2009). This indicates that students' performance on the second problem in the sequence was positively affected by the first problem. In the present study, we are only interested in the first problem in the sequence that a student encountered because it better reflects students' ability to think about integration without the influence of prior knowledge from the other problem.

Findings

In this section, we investigate students' performance on each of the interview problems to see whether or not students had the concept projection of integration that spanned the graphical and algebraic representations in the work-energy context.

Interview 2

The graphical and algebraic problems are presented in Figures 1 and 2. Eleven students were presented the graphical problem first and nine were presented the algebraic problem first. In each problem students had to calculate work done by the spring inside a gun. To do this, students ought to understand that the total work done by a force was determined by accumulating the infinitesimal work dW on infinitesimal distance dx over which the force could be considered constant (so the relation $dW = F.dx$ applied). The accumulation of the infinitesimal work dW could be done by integrating the force function $\int F(x)dx$ when the function was given in algebraic representation or finding the area under the curve of $F(x)$ vs. x when the function was given in graphical representation. This is where the concept of integration comes in and projects itself onto graphical representation (area under the curve of $F(x)$ vs. x) and algebraic representation ($\int F(x)dx$).

Graphical Problem

To calculate work done by the spring force in this problem, instead of finding the area of the curve of $F(x)$ vs. x, students can alternatively find the slope of the graph which is the spring constant "k" and plug it into "$\frac{1}{2}kx^2$" where "x" is the maximum spring compression. Ten out of 11 students who attempted the graphical problem first followed this strategy. Upon being asked to find yet another strategy, six students recognized that the work was the area under the curve of $F(x)$ vs. x after hints were given by the interviewer. Four other students knew that area had some physical meaning but did not know what it was until being explicitly told the meaning by the interviewer. Only one student could spontaneously recognize that work equaled the area under the curve of $F(x)$ vs. x without assistance from the interviewer.

Algebraic Problem

Of the nine students who did the algebraic problem before the graphical problem, only three students spontaneously recognized that work equaled $\int F(x)dx$. The six remaining students attempted to calculate work done by the spring either by finding spring constant $k = \frac{F}{x}$

to plug in "$\frac{1}{2}kx^2$" or by using $W = F.d$ where d was the distance the bullet travelled. Of these six students, three of them recognized that $W = \int F(x)\,dx$ after being provided the hint that force was not constant while the other three did not recognize this relationship until the interviewer explicitly told them about it.

Interview 3

The graphical and algebraic problems in this interview are presented in Figures 3 and 4. There were nine students who attempted the graphical problem first and 11 students who attempted the algebraic problem first. In these problems, students had to calculate work done by the resistance force of a liquid by finding the area under the line of force on the graph of $F(x)$ vs. x or by computing the integral $\int F(x)\,dx$.

Graphical Problem

The difference between the graphical problem in this interview and the one in interview 2 is that in this problem, the only way to find work is to calculate the area under the line of force on the graph of $F(x)$ vs. x, while in the graph problem in interview 2, students can find spring constant "k" which equals the slope of the line and plug in "$\frac{1}{2}kx^2$". Three students spontaneously recognized that work equaled the area under the line. Errors that the other six students made included: finding work using $W = F.d$, finding the slope of the line and using it as "coefficient of friction" and finding the slope and using it as work. Of these six students, three recognized work was calculated from the area under the curve after being given hints by the interviewer, while the other three didn't recognize it until being told explicitly.

Algebraic Problem

Out of 11 students, four spontaneously recognized $W = \int F(x)\,dx$. Errors that the other seven students made included finding work using $W = F.d$ or finding work at two ends and averaging, finding "coefficient of friction" from the algebraic expression of $F(x)$. A few other students said that they knew that work was either derivative or integral but did not know specifically which one. Of these seven students, five recognized it after being hinted by the interviewer, while two did not recognize it until they were explicitly told so by the interviewer.

Interview 4

The graphical and algebraic problems in this interview are presented in Figures 5 and 6. Nine students were presented the graphical problem first and 11 were presented the algebraic problem first. In these problems, students had to calculate work done by the rolling friction force along the curved part of the track by finding the area under the curve of force on the graph of $F(\theta)$ vs. θ together with some unit conversion or by computing $W = \int F(\theta)\,ds = \int F(\theta)\,Rd\theta$. These problems are more difficult than the problems in previous interviews because the area under the curve or integral of the force function $\int F(\theta)\,d\theta$ alone does not yield the value of work done by rolling friction force. Students have to recognize that the given graph is that of force versus angle whereas work is the area under the curve of force versus distance, and the integral of the force function is $\int F(\theta)\,d\theta$ whereas work is $\int F(\theta)\,ds$ in which "$ds = R.d\theta$" is the distance along the circular track of radius R covering the angle $d\theta$. So these problems require students not only to recognize the use of the area under the curve or integral of the algebraic expression but also to understand the process of accumulating infinitesimal work to find the total work. In other words, students must have the concept projection of integration as accumulation in order to solve these problems correctly.

Graphical Problem

Six out of nine students who attempted the graphical problem first easily recognized that they had to find the area under the curve but only one of them spontaneously recognized that the area itself was not the work and knew that he had to convert the unit of the area to the unit of work, while the other five students needed hints to recognize that the area itself was not the value of work. Three other students needed hints to recognize the use of the area under the curve and unit conversion.

Algebraic Problem

All 11 students recognized that they had to integrate the force function, but only one of them spontaneously recognized that he should have $\int F(\theta)\,ds$ instead of $\int F(\theta)\,d\theta$. Five students calculated the integral of force $\int F(\theta)\,d\theta$ and multiplied by the total distance. Five other students just calculated $\int F(\theta)\,d\theta$ and thought it was the value of work. All of these 10 students were able to recognize that they had to either take $\int F(\theta)\,ds$ or convert the unit after taking $\int F(\theta)\,d\theta$ to get the correct value of work after several hints were given by the interviewer.

Conclusions and Implications

Table 1 summarizes the number of students who were able or unable to recognize the appropriate concept projection of integration in the relevant problem contexts presented to them in interviews 2, 3 and 4.

We answer our research question: To what extent does students' concept projection of integration span the context of work-energy problems that provide information in graphical and algebraic representations? From Table 1, we see that most students did not spontaneously recognize the idea that integration was the area under the curve or algebraic integral. This might imply that these students did not have the concept projection of integration spanned over graphical and algebraic representations of functions in the context of work-energy problems. Even when students spontaneously recognized that integration was the area under the curve or algebraic integral, there is evidence suggesting that they might not have the concept projections of integration that span work-energy problems in graphical and algebraic representations. First, only one student in interview 4 spontaneously recognized that the values of area or integral were not the values of work. The rest of the students simply calculated the area or integral without an understanding of the process. Second, there were some students who realized that the area under the curve had some meaning but did not know what the meaning was, and others who knew that they had to either differentiate or integrate the function but did not know which one to do, indicating that most students simply remembered the strategy without understanding the underlying process of integration.

Developing problem solving skills in science often requires an understanding and application of mathematics. This study reveals an interesting pattern underlying students' performance on the use of mathematics in science. Students may not have a deeper understanding of the conceptual underpinning of the mathematical operation even though they can mechanically perform the operation easily. This lack of deeper understanding is revealed when learners were asked to solve non-standard problems that demanded more than an operational knowledge of mathematical procedures. The findings of this study suggest that instructors should facilitate students not only to learn how to perform mathematical operations but also to understand what these operations mean. In other words, instructors must help students build the concept projections for every concept they learn. These concept projections must be supported by complete knowledge bases and span a broad range of contexts and representations. This study is of interest to mathematics and science teachers who attempt to facilitate their students to build strong scientific reasoning on mathematical processes so that they can recognize and use them correctly in science as well as in real-world situations.

References

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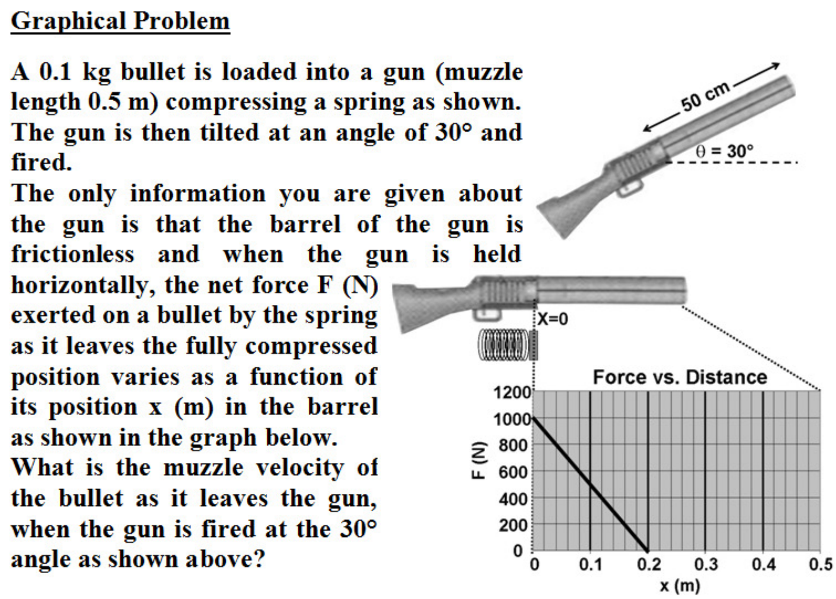
Table 1

Summary of Students' Performance on Integration Tasks in Our Interviews

Interview	***'Area Under Curve'***			***'Algebraic Integral'***		
	Spontaneously Recognize	Recognize After Hints	Do Not Recognize	Spontaneously Recognize	Recognize After Hints	Do Not Recognize
2	1/11	6/11	4/11	3/9	3/9	3/9
3	3/9	3/9	3/9	4/11	5/11	2/11
4	1/9	5/9	3/9	1/11	10/11	0/11

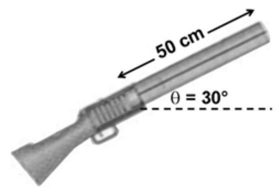


Graphical Problem

A 0.1 kg bullet is loaded into a gun (muzzle length 0.5 m) compressing a spring as shown. The gun is then tilted at an angle of 30° and fired.
The only information you are given about the gun is that the barrel of the gun is frictionless and when the gun is held horizontally, the net force F (N) exerted on a bullet by the spring as it leaves the fully compressed position varies as a function of its position x (m) in the barrel as shown in the graph below.
What is the muzzle velocity of the bullet as it leaves the gun, when the gun is fired at the 30° angle as shown above?

Figure 1. The graphical problem of interview 2.

Algebraic Problem



A 0.1 kg bullet is loaded into a gun (muzzle length 0.5 m) compressing a spring to a maximum of 0.2 m as shown. The gun is then tilted at an angle of 30° and fired.
The only information you are given about the gun is that the barrel of the gun is frictionless and that the gun contains a non-linear spring such that when the gun is held horizontally, the net force F (N) exerted on a bullet by the spring as it leaves the fully compressed position varies as a function of the spring compression x (m) as given by:

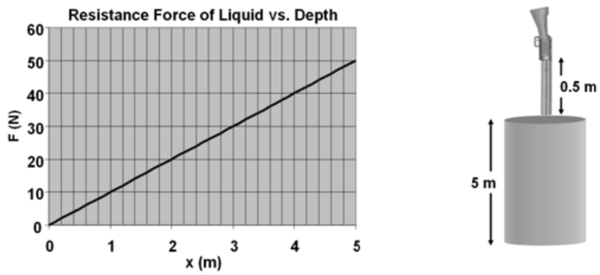
$$F = 1000x + 3000x^2$$

What is the muzzle velocity of the bullet as it leaves the gun, when the gun is fired at the 30° angle as shown above?

Figure 2. The algebraic problem of interview 2.

<u>Graphical Problem</u>

A 0.1 kg bullet is loaded into a gun compressing a spring which has spring constant k = 6000 N/m. The gun is tilted vertically downward and the bullet is fired into a drum 5.0 m deep, filled with a liquid. The barrel of the gun is frictionless. The resistance force provided by the liquid changes with depth as shown in the graph below. The bullet comes to rest at the bottom of the drum

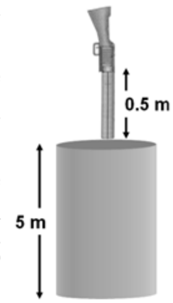


What is the spring compression *x*?

Figure 3. The graphical problem of interview 3.

Algebraic Problem

A 0.1 kg bullet is loaded into a gun compressing a spring which has spring constant k = 6000 N/m. The gun is tilted vertically downward and the bullet is fired into a drum 5.0 m deep, filled with a liquid.
The barrel of the gun is frictionless. The frictional force *F*(N) provided by the liquid changes with depth *x*(m) as per the following function:



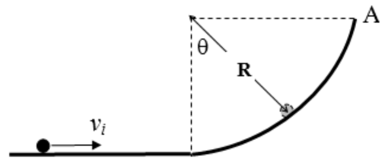
$$F(x) = 10x + 0.6x^2$$

The bullet comes to rest at the bottom of the drum. What is the spring compression *x*?

Figure 4. The algebraic problem of interview 3.

Graphical Problem

A sphere radius r = 1 cm and mass m = 2 kg is rolling at an initial speed v_i of 5 m/s along a track as shown. It hits a curved section (radius R = 1.0 m) and is launched vertically at point A.



The rolling friction on the straight section is negligible.
The magnitude of the rolling friction force acting on the sphere varies as angle θ as per the graph shown below. What is the launch speed of the sphere as it leaves the curve at point A?

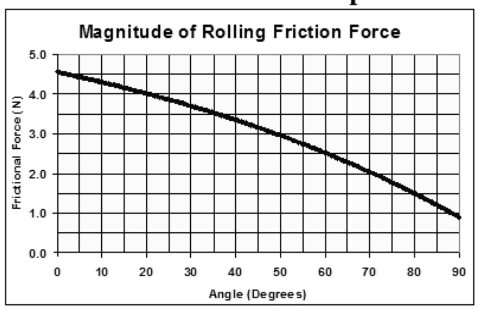
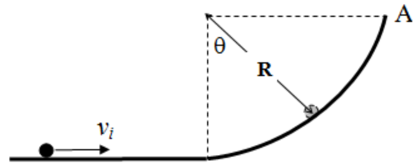


Figure 5. The graphical problem of interview 4.

Algebraic Problem

A sphere radius r = 1 cm and mass m = 2 kg is rolling at an initial speed v_i of 5 m/s along a track as shown. It hits a curved section (radius R = 1.0 m) and is launched vertically at point A. The rolling friction on the straight section is negligible.



The magnitude of the rolling friction force F_{roll} (N) acting on the sphere varies as angle θ (radians) as per the following function

$$F_{roll}(\theta) = -0.7\theta^2 - 1.2\theta + 4.5$$

What is the launch speed of the sphere as it leaves the curve at point A?

Figure 6. The algebraic problem of interview 4.